Attracting current and equilibrium measure for attractors on $\mathbb{P}^k$

Tien-Cuong Dinh

February 2, 2008

Abstract

Let f be a holomorphic endomorphism of $\mathbb{P}^k$ having an attracting set $\mathscr{A}$. We construct an attracting current and an equilibrium measure associated to $\mathscr{A}$. The attracting current is weakly laminar and extremal in the cone of invariant currents. The equilibrium measure is mixing and has maximal entropy.

MSC : 37F, 32H50, 32U40.
Key-words : attracting set, attracting current, structural disc of currents, equilibrium measure, entropy, mixing.

1 Introduction

Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. If S is a smooth positive closed $(1,1)$-form of mass 1 then $d^{-n}(f^n)^*S$ converges weakly to a positive closed $(1,1)$-current T of mass 1 which does not depend on S and has locally Hölder continuous potentials, see Fornæss-Sibony [18, 32]. The current T is totally invariant : $d^{-1}f^*(T) = d^{-k+1}f_*(T) = T$. This is *the Green current associated to* f; its support is *the Julia set*. The self-intersection T^m of T, $1 \leq m \leq k$, is *the Green current of bidegree* (m,m). It satisfies $d^{-m}f^*(T^m) = d^{-k+m}f_*(T^m) = T^m$. We refer to [17, 6, 26, 21, 15] for the basics on the theory of currents and their intersections. We also obtain T^m as the limit in the sense of current of $d^{-mn}(f^n)^*[I]$, where $[I]$ is the current of integration on a generic projective subspace I of dimension $k-m$ [19, 31, 14].

Following Yomdin [34] and Gromov [23], f has topological entropy $k\log d$ and the variational principle [28, 33] implies that all measures, invariant by f, have entropy $\leq k\log d$. The equilibrium measure $\mu := T^k$ of f is the unique invariant probability measure of maximal entropy $k\log d$, see Fornæss-Sibony [19] and Briend-Duval [5]. The main part of the dynamics is, in some sense, concentrated on the support of μ. In this paper, we study the dynamics on some

attracting sets of f. They are disjoint from $\mathrm{supp}(\mu)$ or disjoint from $\mathrm{supp}(T^m)$ for some m.

A compact subset $\mathscr{A}$ is an *attracting set* for f if it admits a neighbourhood U such that $f(U) \Subset U$ and $\mathscr{A} = \cap_{n\geq 0} f^n(U)$. Attracting sets are invariant : $f(\mathscr{A}) = \mathscr{A}$. If moreover f admits a dense orbit in $\mathscr{A}$ then $\mathscr{A}$ is said to be *an attractor*. We refer to [22, 24, 20, 30] for basic properties and examples of attracting sets and attractors, and to [8] for the study of weakly saddle measures outside $\mathrm{supp}(\mu)$.

We now describe the class of attracting sets that we study here. This class contains the examples studied in the previous references [1]. Let I and L be two projective subspaces of $\mathbb{P}^k$, of dimensions $p-1$ and $k-p$ respectively, $1 \leq p \leq k-1$. We assume that $I \cap U = \varnothing$ and that L is contained in U. In particular, L does not intersect I. Note that in our study the case $p = k$ is not interesting. Indeed, f contracts the Kobayashi pseudo-distance, which is a distance in this case, hence $\mathscr{A}$ is reduced to a point, see also [32].

Let $\pi : \mathbb{P}^k \setminus I \to L$ be the canonical projection of center I. More precisely, if $I(x)$ is the projective space, of dimension p, containing I and passing through a point $x \in \mathbb{P}^k \setminus I$ then $\pi(x)$ is the unique intersection point of L with $I(x)$. We consider the point $\pi(x)$ as the origin of the complex vector space $I(x) \setminus I \simeq \mathbb{C}^p$, where I is viewed as the hyperplane at infinity of $I(x) \simeq \mathbb{P}^p$. In other words, $\mathbb{P}^k \setminus I$ is considered as a vector bundle over L. If $x \in L$ we have $\pi(x) = x$. Our main hypothesis is

the open set $U \cap I(x)$ in $I(x) \setminus I \simeq \mathbb{C}^p$ is star-shaped at x for every $x \in L$.

In particular, U and $\mathbb{P}^k \setminus U$ are connected. Observe that the previous geometric condition is often easy to check and is stable under small pertubations on f. If I is a point and L is a projective hyperplane (i.e. $p = 1$), the previous hypothesis is equivalent to the property that the open subset $\mathbb{P}^k \setminus U$ of $\mathbb{P}^k \setminus L \simeq \mathbb{C}^k$ is star-shaped at I. In particular, this last property is satisfied when $\mathbb{P}^k \setminus U$ is convex.

Consider a generic subspace I' of dimension $p-1$ close enough to I. Then $d^{-(k-p+1)n}(f^n)^*[I']$ converges to T^{k-p+1}. Since $d^{-(k-p+1)n}(f^n)^*[I']$ is supported on $\mathbb{P}^k \setminus U$, the support of T^{k-p+1} does not intersect U. In particular, we have $\mathrm{supp}(T^{k-p+1}) \cap \mathscr{A} = \varnothing$, hence $\mathrm{supp}(\mu) \cap \mathscr{A} = \varnothing$. Studying the dynamics on $\mathscr{A}$ allows us to understand the dynamics outside the support of μ. We will associate to $\mathscr{A}$ a dynamically interesting invariant current and measure.

The positive closed current $[L]$ of integration on L is of bidegree (p, p) and of bidimension $(k-p, k-p)$. It has compact support in U. Using a standard convolution we can construct smooth positive closed (p, p)-forms with compact support in U, see also Section 3.

[1] For attracting sets and attractors in a weaker sense, see [3] and the references therein

Theorem 1.1. *Let f, I and U be as above. Let R be a continuous positive closed (p,p)-form of mass 1 with compact support in U. Then $d^{-(k-p)n}(f^n)_*R$ converges weakly to a positive closed current τ of mass 1 with support on $\mathscr{A}$. The current τ does not depend on R. Moreover, it is weakly laminar, invariant ($f_*\tau = d^{k-p}\tau$) and is extremal in the cone of invariant positive closed (p,p)-currents with support in $\mathscr{A}$.*

We call τ *the attracting current associated to $\mathscr{A}$*. The hypothesis on the geometry of U is necessary. For this, we can consider the map $f : \mathbb{P}^2 \to \mathbb{P}^2$, $f[z_0 : z_1 : z_2] = [z_0^d : z_1^d : z_2^d]$ and $\mathbb{P}^2 \setminus U := \{\frac{1}{2}|z_i| \leq |z_j| \leq 2|z_i|, \text{ for } 0 \leq i,j \leq 2\}$. Then $\mathscr{A} = \{z_0 = 0\} \cup \{z_1 = 0\} \cup \{z_2 = 0\}$. We can see that $\mathscr{A}$ admits three "attracting" currents (currents of integration on $\{z_i = 0\}$). This property is stable under small pertubations of f.

There are obvious situation where Theorem 1.1 applies. Let f be a polynomial map in $\mathbb{C}^k$ of algebraic degree $d > 1$ with holomorphic extension to $\mathbb{P}^k$, then the hyperplane at infinity is attracting and Theorem 1.1 can be applied to small pertubations of f.

Recall here that the projective space $\mathbb{P}^k$ is endowed with the Fubini-Study form ω_{FS}. This is a Kähler form normalized by $\int \omega_{\mathrm{FS}}^k = 1$. *The mass* of a positive closed (p,p)-current R on an open set W is defined by $\|R\|_W := \int_W R \wedge \omega_{\mathrm{FS}}^{k-p}$. The mass $\|R\| := \|R\|_{\mathbb{P}^k}$ of R (on $\mathbb{P}^k$) depends only on its cohomology class in $H^{p,p}(\mathbb{P}^k, \mathbb{C}) \simeq \mathbb{C}$. In particular, if R is of mass 1, it is cohomologous to ω_{FS}^p. Then $\|(f^n)^*R\| = \|(f^n)^*\omega_{\mathrm{FS}}^p\| = d^{pn}$ and

$$\begin{aligned}
\|(f^n)_*R\| &= \int (f^n)_*R \wedge \omega_{\mathrm{FS}}^{k-p} = \int R \wedge (f^n)^*\omega_{\mathrm{FS}}^{k-p} \\
&= \int \omega_{\mathrm{FS}}^p \wedge (f^n)^*\omega_{\mathrm{FS}}^{k-p} = \|(f^n)^*\omega_{\mathrm{FS}}^{k-p}\| = d^{(k-p)n}.
\end{aligned}$$

The current τ is *weakly laminar or web-like*, see [9], if it can be decomposed into currents of integration on complex manifolds of dimension $k-p$ as follows

i) There exists a family of complex manifolds $V \subset \mathbb{P}^k$ of dimension $k-p$, not necessarily closed in $\mathbb{P}^k$, with finite volume. We denote by $[V]$ the current of integration on V, it is positive but not necessarily closed.

ii) There is a positive measure λ defined on the family of these currents $[V]$ such that

$$\int \mathrm{volume}(V)\mathrm{d}\lambda(V) < \infty \qquad \text{and} \qquad \tau = \int [V]\mathrm{d}\lambda(V).$$

More precisely, if Φ is a test smooth $(k-p,k-p)$-form then

$$\langle \tau, \Phi \rangle = \int \left(\int_V \Phi \right) \mathrm{d}\lambda(V).$$

Theorem 1.1 implies that $\mathscr{A}$ contains pieces of complex manifolds of dimension $k-p$. This is an important property which allows to study the dynamics on $\mathscr{A}$ using geometric methods, see [1, 16, 8] and Section 6.

Following an idea of Sibony, see e.g. [32] and [2] for historical comments, we define *the equilibrium measure associated to* $\mathscr{A}$, as intersection of invariant currents, by

$$\nu := T^{k-p} \wedge \tau.$$

This is a probability measure invariant by f ($f_*(\nu) = \nu$) and supported on $\mathscr{A}$. Since T has locally Hölder continuous potentials, the measure ν has positive Hausdorff dimension and has no mass on points, see [5, 32, 13].

Using an idea of de Thélin [7], we prove in Section 2 that the topological entropy of f on U is bounded by $(k-p)\log d$. The variational principle [28, 33] implies that any invariant measure with support in $\mathscr{A}$, in particular ν, has entropy $\leq (k-p)\log d$. We have the following theorem.

Theorem 1.2. *Under the above notation, the measure ν has maximal entropy $(k-p)\log d$ and is mixing.*

Note that we can associate to $\mathscr{A}$ the invariant currents $T^m \wedge \tau$, $1 \leq m \leq k-p-1$, which should be useful in the study of the dynamics on $\mathscr{A}$.

The precise outline of the paper is as follows. In Section 2, we generalize a theorem due to de Thélin to the case of higher dimension. In Section 3, we introduce the main tools needed in the construction of the attracting current. We use here a new method introduced by Nessim Sibony and the author in [15] to deal with currents of bidegree (p,p). Theorems 1.1 and 1.2 are proved in Sections 4 and 5. Further remarks and questions are given in Section 6. Appendix A contains some useful results on the theory of slicing of currents. Throughout the paper, $\mathscr{C}(U)$ denotes the set of positive closed currents of mass 1 with compact support in U and $\mathscr{D}$ denotes the set of all the limit values of $d^{-(k-p)n}(f^n)_*R_n$ with $R_n \in \mathscr{C}(U)$.

2 Entropy

In this section we give a sharp upper bound for the topological entropy of f outside the support of T^m. We follow an idea due to de Thélin in [7] where he studied the case of dimension 2.

Consider a subset W of $\mathbb{P}^k$, not necessarily invariant. A family F of points in W is said to be *(n,ϵ)-separated*, $\epsilon > 0$, if for all distinct points x and y in F, there is an integer j, $0 \leq j \leq n-1$, such that distance$(f^j(x), f^j(y)) > \epsilon$. Define the *topological entropy* of f on W by

$$h_t(f,W) := \sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log\max\big\{\#F,\ F\subset W\ (n,\epsilon)\text{-separated}\big\},$$

see [4, 7]. By Yomdin [34] and Gromov [23], the topological entropy $h_t(f) := h_t(f, \mathbb{P}^k)$ of f is equal to $k \log d$, see also [11, 12]. The main result of this section is the following theorem, see [7] for the case $k = m = 2$.

Theorem 2.1. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Let T be the Green current of f. If W is a subset of $\mathbb{P}^k$ such that $\overline{W} \cap \operatorname{supp}(T^m) = \varnothing$, $1 \leq m \leq k$, then*

$$h_t(f, W) \leq (m-1) \log d.$$

Proof. The proof is based on an idea of Gromov [23] and on the speed of convergence toward the Green current by Fornæss-Sibony [18, 32]. Fix an open neighbourhood W' of $\overline{W}$ such that $W' \Subset \mathbb{P}^k \setminus \operatorname{supp}(T^m)$.

Let $\Gamma_{[n]}$ denote the graph of the map $(f, f^2, \ldots, f^{n-1})$ in $(\mathbb{P}^k)^n$, i.e. the set of points

$$(x, f(x), \ldots, f^{n-1}(x)), \quad x \in \mathbb{P}^k.$$

The projection Π on the first factor $\mathbb{P}^k$ defines a biholomorphic map between $\Gamma_{[n]}$ and $\mathbb{P}^k$. Consider an (n, ϵ)-separated subset $F \subset W$. We associate to F the family of balls in $(\mathbb{P}^k)^n$ of center $(x, f(x), \ldots, f^{n-1}(x))$, with $x \in F$, and of radius $\epsilon/2$. Here, the distance between two points in $(\mathbb{P}^k)^n$ is the maximum of the distances between their projections on the factors $\mathbb{P}^k$. Since F is (n, ϵ)-separated, these balls are disjoint. They are contained in $\Pi^{-1}(W')$ when ϵ is small enough.

An inequality of Lelong [29] implies that the volume of $\Gamma_{[n]}$ in each ball is larger than $c\epsilon^{2k}$ for some constant $c > 0$. We use here the volume with respect to the metric induced by the Kähler form $\sum \Pi_i^*(\omega_{\mathrm{FS}})$, where Π_i denotes the canonical projection of $(\mathbb{P}^k)^n$ on its i-th factor. We deduce an estimate of the number of balls and obtain the following Gromov's inequality

$$h_t(f, W) \leq \operatorname{lov}(f, W') := \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{volume}(\Pi^{-1}(W') \cap \Gamma_{[n]}).$$

We will show that $\operatorname{volume}(\Pi^{-1}(W') \cap \Gamma_{[n]}) \lesssim n^k d^{(m-1)n}$ which implies the theorem. Since $\Gamma_{[n]}$ has dimension k and since Π defines a biholomorphic map between $\Gamma_{[n]}$ and $\mathbb{P}^k$, we have

$$\begin{aligned}
\operatorname{volume}(\Pi^{-1}(W') \cap \Gamma_{[n]}) &= \frac{1}{k!} \int_{\Pi^{-1}(W') \cap \Gamma_{[n]}} \Big(\sum_{i=0}^{n-1} \Pi_i^*(\omega_{\mathrm{FS}}) \Big)^k \\
&= \frac{1}{k!} \int_{W'} \Big(\sum_{i=0}^{n-1} (f^i)^*(\omega_{\mathrm{FS}}) \Big)^k \\
&= \frac{1}{k!} \sum_{0 \leq n_i \leq n-1} \int_{W'} (f^{n_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_k})^*(\omega_{\mathrm{FS}}).
\end{aligned}$$

Hence, it is enough to show for $0 \leq n_i \leq n$ that

$$\int_{W'} (f^{n_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_k})^*(\omega_{\mathrm{FS}}) \lesssim d^{(m-1)n}.$$

To this end, we prove by induction on (r, s), $0 \leq r \leq m$ and $0 \leq s \leq k - m + r$, that

$$\|T^{m-r} \wedge (f^{n_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}})\|_{W_{r,s}} \leq c_{r,s} d^{n(r-1)},$$

where $W_{r,s}$ is a neighbourhood of $\overline{W}'$ and $c_{r,s} \geq 0$ is a constant independent of n. We obtain the result by taking $r = m$ and $s = k$.

It is clear that the previous inequality holds for $r = 0$ or $s = 0$ with $W_{r,s} = \mathbb{P}^k \setminus \mathrm{supp}(T^m)$ and $c_{r,s} = 1$. Assume this inequality for the cases of $(r-1, s-1)$ and $(r, s-1)$. Let $W_{r,s}$ be a neighbourhood of $\overline{W}'$ strictly contained in $W_{r-1,s-1}$ and $W_{r,s-1}$. Let $\chi \geq 0$ be a smooth cut-off function with support in $W_{r-1,s-1} \cap W_{r,s-1}$ which is equal to 1 on $W_{r,s}$. We only have to prove that

$$\int T^{m-r} \wedge (f^{n_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge \chi \omega_{\mathrm{FS}}^{k-m+r-s} \leq c_{r,s} d^{n(r-1)}.$$

Since T and ω_{FS} have the same mass, they belong to the same cohomology class in $H^{1,1}(\mathbb{P}^k, \mathbb{C})$. Moreover, T has locally continuous potentials. Hence, there is a continuous quasi-psh function g such that $\omega_{\mathrm{FS}} = T - dd^c g$. We have

$$(f^{n_1})^* \omega_{\mathrm{FS}} = d^{n_1} T - dd^c (g \circ f^{n_1}).$$

The integral that we have to estimate, is equal to the sum of the following integrals

$$d^{n_1} \int T^{m-r+1} \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge \chi \omega_{\mathrm{FS}}^{k-m+r-s}$$

and

$$-\int T^{m-r} \wedge dd^c (g \circ f^{n_1}) \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge \chi \omega_{\mathrm{FS}}^{k-m+r-s}.$$

Using the case of $(r-1, s-1)$ we can bound the first integral by $cd^{n(r-1)}$. Stokes' theorem implies that second integral is equal to

$$-\int T^{m-r} \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge (g \circ f^{n_1}) dd^c \chi \wedge \omega_{\mathrm{FS}}^{k-m+r-s}$$

which is bounded by

$$\|g\|_\infty \|\chi\|_{\mathscr{C}^2} \|T^{m-r} \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}})\|_{W_{r,s-1}}$$

since χ has support in $W_{r,s-1}$. We obtain the result using the case of $(r, s-1)$. $\square$

From the variational principle [28, 33], we deduce the following result.

Corollary 2.2. *Let ν be a probability measure invariant by f. Assume that ν has compact support in $\mathbb{P}^k \setminus \mathrm{supp}(T^m)$. Then the entropy of ν satisfies*

$$h(\nu) \leq (m-1) \log d.$$

We refer to [28, 33] for the definition of the entropy of an invariant measure.

3 Structure of the cone of currents

In this section, we introduce the main tools that we use in order to construct the attracting current. The approach is similar to the one in [15]. Let $\mathscr{C}(U)$ denote the set of positive closed (p,p)-currents R of mass 1 in $\mathbb{P}^k$ with compact support in the open set U. Consider a real smooth $(k-p,k-p)$-form Φ on U, not necessarily with compact support, such that $\mathrm{dd^c}\Phi \geq 0$. Define the function Λ_Φ on $\mathscr{C}(U)$ by $\Lambda_\Phi(R) := \langle R, \Phi\rangle$. These Λ_Φ play the role of *psh functions* on $\mathscr{C}(U)$, see Lemma 3.2 below. The following lemma implies that they separate the currents in $\mathscr{C}(U)$. Recall that $\overline{U}$ does not intersect the space I of dimension $p-1$.

Lemma 3.1. *Let Φ be a real smooth $(k-p,k-p)$-form with compact support in U. Then there are smooth $(k-p,k-p)$-forms Φ^+, Φ^-, not necessarily with compact support, such that $\mathrm{dd^c}\Phi^\pm \geq 0$ and $\Phi = \Phi^+ - \Phi^-$. Moreover, there exists a real $(k-p,k-p)$-form Ψ on U such that $\mathrm{dd^c}\Psi$ is strictly positive.*

Proof. If Ψ is as above, then $\Phi^+ := \Phi + c\Psi$ and $\Phi^- := c\Psi$, $c>0$ large enough, satisfy the lemma. We only have to construct such a form Ψ. Fix a point x in $\overline{U}$. It is enough to construct a form Ψ_x such that $\mathrm{dd^c}\Psi_x \geq 0$ in a neighbourhood of $\overline{U}$, and $\mathrm{dd^c}\Psi_x$ strictly positive at x. We obtain Ψ by taking a finite sum of such forms Ψ_x.

Let φ be a smooth strictly psh function on $I(x)\setminus I \simeq \mathbb{C}^p$, see the Introduction. Then $\varphi[I(x)]$ defines a $(k-p,k-p)$-current Ψ'_x on $\mathbb{P}^k \setminus I$ such that $\mathrm{dd^c}\Psi'_x \geq 0$ and $\mathrm{dd^c}\Psi'_x$ does not vanish at x. We use here the standard method to regularize Ψ'_x.

Let $\mathrm{Aut}(\mathbb{P}^k)$ be the group of holomorphic automorphisms of $\mathbb{P}^k$. This is a complex Lie group of dimension k^2+2k. Let ρ be a smooth probability measure on $\mathrm{Aut}(\mathbb{P}^k)$ with support in a small neighbourhood of the identity map $\mathrm{id} \in \mathrm{Aut}(\mathbb{P}^k)$. Define

$$\Psi''_x := \int A_*(\Psi'_x)\mathrm{d}\rho(A).$$

Then Ψ''_x is smooth and is defined out of a small neighbourhood of I. We also have

$$\mathrm{dd^c}\Psi''_x := \int A_*(\mathrm{dd^c}\Psi'_x)\mathrm{d}\rho(A).$$

Hence, $\mathrm{dd^c}\Psi''_x$ is positive and x is very close to the support of $\mathrm{dd^c}\Psi''_x$. Replacing Ψ''_x by $A^*(\Psi''_x)$, where A is a suitable automorphism close to the identity, we can assume that $\mathrm{dd^c}\Psi''_x$ does not vanish at x. We can choose a finite number of automorphisms A_i, close to the identity, which fix the point x, such that if

$$\Psi_x := \sum A_i^*(\Psi''_x)$$

then $\mathrm{dd^c}\Psi_x$ is strictly positive at x. $\square$

Let V be a complex manifold of dimension m. Let $\pi_V : V \times U \to V$ and $\pi_U : V \times U \to U$ be the canonical projections. Consider a positive closed (p,p)-current $\mathscr{R}$ in $V \times U$ such that $\pi_U(\mathrm{supp}(\mathscr{R})) \Subset U$. We say that $\mathscr{R}$ is *horizontal*, see also [15]. It is shown in Proposition A.1 in Appendix A that the slice $\langle \mathscr{R}, \pi_V, \theta \rangle$ exists for every $\theta \in V$. This is a positive closed (p,p)-current with compact support in $\{\theta\} \times U$. We often identify it to a (p,p)-current R_θ in U. The mass of R_θ does not depend on θ, see Proposition A.3. We assume that this mass is equal to 1. Then one obtains a map $h : \theta \mapsto R_\theta$ from V into $\mathscr{C}(U)$. We say that the map h or the family (R_θ) defines a *structural variety* of $\mathscr{C}(U)$. We deduce from Remark A.2 the following lemma.

Lemma 3.2. *Let $(R_\theta)_{\theta \in V}$ be a structural variety as above. Let Φ be a real smooth $(k-p, k-p)$-form on U such that $\mathrm{dd^c}\Phi \geq 0$. Then $\theta \mapsto \langle R_\theta, \Phi \rangle$ defines a psh function on $\theta \in V$.*

In what follows we construct and use only some special structural discs (R_θ) such that R_θ depends continuously on θ, where V is a holomorphic disc in $\mathbb{C}$. The construction of these discs is as follows. Fix a chart W of $\mathrm{Aut}(\mathbb{P}^k)$ containing $\mathrm{id} \in \mathrm{Aut}(\mathbb{P}^k)$ and local holomorphic coordinates y, $\|y\| < 1$, such that $y = 0$ at id. Then we can consider the scalar multiplication $\lambda_\theta(y) := (1-\theta)y$ for $\theta \in \mathbb{C}$ and $\|y\| < \min(1, |1-\theta|^{-1})$. Fix a small neighbourhood $W' \Subset W$ of id. Sometimes we identify y with the automorphism it represents.

We define a holomorphic self-map A_θ on $\mathbb{P}^k \setminus I$. If x is a point in $\mathbb{P}^k \setminus I$, it belongs to $I(x) \setminus I$ and we consider $I(x) \setminus I$ as a complex vector space with origin at $\pi(x) = I(x) \cap L$, see the Introduction. We define $A_\theta(x)$ as the multiplication, in $I(x) \setminus I$, of x by θ. If $\theta = 0$ then A_0 is equal to the projection π of center I onto L. Otherwise A_θ can be extended to an automorphism on $\mathbb{P}^k$ which fixes the points in I. In particular, we have $A_1 = \mathrm{id}$.

Let $U' \Subset U$ be an open set such that $f(U) \Subset U'$. Let V be a simply connected neighbourhood of the interval $[0,1]$ in $\mathbb{C}$. We choose V and W' small enough so that $\lambda_\theta(A) \circ A_\theta(U') \Subset U$ for every $A \in \overline{W}'$ and $\theta \in \overline{V}$. Here we use the geometric assumption on U (see the Introduction), which implies this property for $\theta \in [0,1]$.

Consider $R \in \mathscr{C}(U')$ and an automorphism $A \in W'$. In order to get the picture, the reader can consider the case where R and the currents $R_{A,\theta}$, $\mathscr{R}_A$ below, are defined by integration on complex manifolds. The currents $R_{A,\theta}$, $\theta \in V$, are obtained as images of R under a holomorphic family of maps. More precisely, define $R_{A,\theta} := \lambda_\theta(A)_*(A_\theta)_*(R)$ for $\theta \in V$. These currents belong to $\mathscr{C}(U)$. It is clear that $R_{A,\theta}$ depends continuously on $\theta \in V$. We show that $(R_{A,\theta})$ defines a structural disc in $\mathscr{C}(U)$, i.e. they are slices of some current $\mathscr{R}_A$. But first observe that $R_{A,1} = R$, and $R_A := R_{A,0}$ is independent of R, and is defined by integration on the projective space $A(L)$. In other words, the structural disc $(R_{A,\theta})$, which contains R, passes through a fixed point R_A when R varies.

Define the meromorphic map $\Lambda_A : V \times \mathbb{P}^k \to \mathbb{P}^k$ by

$$\Lambda_A(\theta, x) := A_\theta^{-1}\lambda_\theta(A)^{-1}(x)$$

which is locally a holomorphic submersion outside $\{0\} \times \mathbb{P}^k$. Then the positive closed (p,p)-current $\mathscr{R}_A := \Lambda_A^*(R)$ is well defined in $\Lambda_A^{-1}(U)$ which is an open set in $(V \setminus \{0\}) \times U$. Moreover, when $\theta \to 0$, the support of $\Lambda_A^{-1}(U)$ clusters only on the set $\{0\} \times A(L)$. Then $\mathscr{R}_A$ is a positive closed current on $(V \times U) \setminus (\{0\} \times A(L))$. Since the dimension of $\{0\} \times A(L)$ is strictly smaller than the dimension of $\mathscr{R}_A$, the trivial extension of $\mathscr{R}_A$ across $\{0\} \times A(L)$ is positive and closed [25]. It is easy to check that the slices $\langle \mathscr{R}_A, \pi_V, \theta\rangle$ are equal to $R_{A,\theta}$ for $\theta \neq 0$. This family of currents is continuous and converges to $R_{A,0}$ as $\theta \to 0$. We deduce that $\langle \mathscr{R}_A, \pi_V, \theta\rangle = R_{A,\theta}$ for every θ, see Appendix A below, and that $(R_{A,\theta})$ defines a continuous structural disc in $\mathscr{C}(U)$.

We now introduce a smoothing. Let ρ be a smooth positive probability measure with compact support in $\mathrm{Aut}(\mathbb{P}^k)$. If R is a current in $\mathbb{P}^k$ then the current $\int A_*(R)\mathrm{d}\rho(A)$, $A \in \mathrm{Aut}(\mathbb{P}^k)$, is smooth and converges to R when ρ tends to the Dirac mass at $\mathrm{id} \in \mathrm{Aut}(\mathbb{P}^k)$. Fix ρ with support in W'. Define

$$R_\theta := \int R_{A,\theta}\mathrm{d}\rho(A).$$

It is clear that (R_θ) is also a continuous structural disc in $\mathscr{C}(U)$. They are slices of the current $\mathscr{R} := \int \mathscr{R}_A \mathrm{d}\rho(A)$. We also have $R_1 = R$ and R_0 independent of R.

Proposition 3.3. *The current R_θ is smooth for every $\theta \in V \setminus \{1\}$. Moreover there exist constants $r > 0$, $c > 0$ and $m > 0$, independent of R, such that*

1. *$\|R_\theta - R_0\|_\infty \leq c|\theta|$ for $|\theta| < r$;*
2. *If R is a continuous form and $M(R, \cdot)$ denotes the modulus of continuity of R, then $\|R_\theta - R\|_\infty \leq c(|\theta - 1|\|R\|_\infty + M(R, m|\theta - 1|))$ for $|\theta - 1| < r$.*

Proof. We have $R_\theta = \int A_*(A_\theta)_* R\mathrm{d}(\lambda_\theta)_*\rho(A)$. For $\theta \neq 1$, since $(\lambda_\theta)_*\rho$ is a smooth function, R_θ is smooth.

1. Let Φ be a smooth $(k-p, k-p)$-form with compact support in U. One needs to show that $|\langle R_\theta - R_0, \Phi\rangle| \leq c|\theta|\|\Phi\|_{\mathscr{L}^1}$. Using the identity $R_\theta = \int A_*(A_\theta)_* R\mathrm{d}(\lambda_\theta)_*\rho(A)$, we get

$$\langle R_\theta, \Phi\rangle = \langle R, \Phi'_\theta\rangle, \quad \text{where} \quad \Phi'_\theta := (A_\theta)^* \int A^*\Phi\mathrm{d}(\lambda_\theta)_*\rho(A).$$

Hence, $\langle R_\theta - R_0, \Phi\rangle = \langle R, \Phi'_\theta - \Phi'_0\rangle$ and

$$\begin{aligned}
\Phi'_\theta - \Phi'_0 &:= (A_\theta)^* \int A^*\Phi\mathrm{d}(\lambda_\theta)_*\rho(A) - (A_0)^* \int A^*\Phi\mathrm{d}\rho(A) \\
&= (A_\theta)^* \int A^*\Phi\mathrm{d}\left((\lambda_\theta)_*\rho - \rho\right)(A) \\
&\quad + (A_\theta)^* \int A^*\Phi\mathrm{d}\rho(A) - (A_0)^* \int A^*\Phi\mathrm{d}\rho(A).
\end{aligned}$$

The form $\Phi'_\theta - \Phi'_0$ is defined and smooth in $\mathbb{P}^k \setminus I$. We only have to check that $\|\Phi'_\theta - \Phi'_0\|_{U,\infty} \leq c|\theta|\|\Phi\|_{\mathscr{L}^1}$, i.e. to bound the last sum.

Since $\|(\lambda_\theta)_*\rho - \rho\|_{\mathscr{C}^1} \leq \text{const}|\theta|$, the first integral in the sum above defines a form with $\mathscr{C}^1$-norm bounded by $\text{const}|\theta|\|\Phi\|_{\mathscr{L}^1}$. It follows that the first term of the sum is bounded by $\text{const}|\theta|\|\Phi\|_{\mathscr{L}^1}$. The last two integrals define a form with $\mathscr{C}^1$-norm bounded by $\text{const}\|\Phi\|_{\mathscr{L}^1}$. We deduce that the rest of the sum is also bounded by $\text{const}|\theta|\|\Phi\|_{\mathscr{L}^1}$ since $\|A_\theta - A_0\|_{\mathscr{C}^1(U)} \leq \text{const}|\theta|$.

2. There exists $m > 0$ such that when $|\theta - 1| < r$, r small enough, and $A \in W'$ we have $\|\lambda_\theta(A) \circ A_\theta - \text{id}\|_{\mathscr{C}^1} \leq m|\theta - 1|$. Hence,

$$\|R_{A,\theta} - R\|_\infty \leq \text{const}(|\theta - 1|\|R\|_\infty + M(R, m|\theta - 1|)).$$

The estimate for R_θ follows from its definition. $\square$

4 Attracting current

In this section we prove Theorem 1.1. Let $\mathscr{D}$ denote the set of all the currents S obtained as limit values of $d^{-(k-p)n}(f^n)_*R_n$ with R_n in $\mathscr{C}(U)$. Observe that such a current S has support in $\mathscr{A}$ and there exist $S_n \in \mathscr{D}$ such that $S = d^{-(k-p)n}(f^n)_*S_n$. Indeed, if $S = \lim d^{-(k-p)n_i}(f^{n_i})_*R_{n_i}$ and if S_n is a limit value of $d^{-(k-p)(n_i-n)}(f^{n_i-n})_*R_{n_i}$, $i \to \infty$, then $S = d^{-(k-p)n}(f^n)_*S_n$. In particular, $\mathscr{D}$ is a convex compact set.

Proposition 4.1. *Let Φ be a real smooth $(k-p, k-p)$-form on U such that $\mathrm{dd^c}\Phi \geq 0$. Then there exists a constant c_Φ such that $\langle S, \Phi\rangle \leq c_\Phi$ for every $S \in \mathscr{D}$ and if R is a continuous form in $\mathscr{C}(U')$, then*

$$\lim_{n\to\infty} \langle d^{-(k-p)n}(f^n)_*R, \Phi\rangle = c_\Phi.$$

In particular, if $\mathrm{dd^c}\Phi = 0$ we have $\langle S, \Phi\rangle = c_\Phi$ for every $S \in \mathscr{D}$.

Proof. The case where $\Phi = \omega_{\text{FS}}^{k-p}$ is trivial since $\langle d^{-(k-p)n}(f^n)_*R_n, \Phi\rangle = 1$. Replacing Φ by $A\Phi + \frac{1}{2}\omega_{\text{FS}}^{k-p}$, $A > 0$, we can assume that $0 \leq \Phi \leq \omega_{\text{FS}}^{k-p}$ on U'. Then for $n \geq 1$ we have $\|d^{-(k-p)n}(f^n)^*\Phi\|_U \leq \|d^{-(k-p)n}(f^n)^*\omega_{\text{FS}}^{k-p}\| = 1$ and $\mathrm{dd^c} d^{-(k-p)n}(f^n)^*\Phi \geq 0$ on U.

Let (i_n) be a sequence of integers, $i_n > n$, and R'_{i_n} be positive closed (p,p)-currents of mass 1 with support in U such that $\langle d^{-(k-p)i_n}(f^{i_n})_*R'_{i_n}, \Phi\rangle$ converges to a constant c_Φ. We choose i_n and R'_{i_n} so that c_Φ is the maximal value that we can obtain in this way. It is clear that c_Φ satisfies the inequality in the proposition. Since $0 \leq \Phi \leq \omega_{\text{FS}}^{k-p}$, we have $0 \leq c_\Phi \leq 1$. Define $R_n := d^{-(k-p)(i_n-n)}(f^{i_n-n})_*(R'_{i_n})$. Then $\lim\langle d^{-(k-p)n}(f^n)_*R_n, \Phi\rangle = c_\Phi$. All the currents R_n are supported in U'.

Define the structural discs $(R_{n,\theta})$ associated to R_n as in the last section. We have

$$\varphi_n(\theta) := \langle d^{-(k-p)n}(f^n)_* R_{n,\theta}, \Phi\rangle = \langle R_{n,\theta}, d^{-(k-p)n}(f^n)^*\Phi\rangle.$$

Since $R_{n,\theta}$ depends continuously on θ and since $d^{-(k-p)n}(f^n)^*\Phi$ is smooth, $\varphi_n(\theta)$ is a continuous function. Lemma 3.2, applied to $R_{n,\theta}$ and to $d^{-(k-p)n}(f^n)^*\Phi$, implies that $\varphi_n(\theta)$ is subharmonic on $\theta \in V$. On the other hand, we have $\limsup \varphi_n \leq c_\Phi$ (by definition of c_Φ) and

$$\lim_{n\to\infty} \varphi_n(1) = \lim_{n\to\infty} \langle d^{-(k-p)n}(f^n)_* R_n, \Phi\rangle = c_\Phi.$$

By maximum principle and Hartogs lemma, φ_n converges in $\mathscr{L}^1_{loc}(V)$ to c_Φ, see [27]. Proposition 3.3 and the inequality $\|d^{-(k-p)n}(f^n)^*\Phi\|_U \leq 1$ imply that $|\varphi_n(\theta) - \varphi_n(0)| \leq c|\theta|$. We then deduce that $\lim \varphi_n(0) = c_\Phi$.

Consider now a continuous form $R \in \mathscr{C}(U')$. Define, as in the last section, the structural disc (R_θ) associated to R and continuous subharmonic functions

$$\psi_n(\theta) := \langle d^{-(k-p)n}(f^n)_* R_\theta, \Phi\rangle = \langle R_\theta, d^{-(k-p)n}(f^n)^*\Phi\rangle.$$

Since $R_0 = R_{n,0}$ we have $\psi_n(0) = \varphi_n(0)$. Hence, $\lim \psi_n(0) = c_\Phi$. This, combined with the inequality $\limsup \psi_n \leq c_\Phi$, implies that $\psi_n \to c_\Phi$ in $\mathscr{L}^1_{loc}(V)$. On the other hand, Proposition 3.3 gives that $\lim_{\theta\to 1}(\sup_n |\psi_n(\theta) - \psi_n(1)|) = 0$. Therefore, $\lim \psi_n(1) = c_\Phi$. Since $R_1 = R$, we obtain $\lim\langle d^{-(k-p)n}(f^n)_* R, \Phi\rangle = c_\Phi$.

When $dd^c\Phi = 0$, the inequality in the proposition applied to $\pm\Phi$ yields $\langle S, \Phi\rangle = c_\Phi$ for every $S \in \mathscr{D}$. □

The last proposition and Lemma 3.1 show that if R is a continuous positive closed (p,p)-form of mass 1 with support in U then $d^{-(k-p)n}(f^n)_* R$ converges to a current τ which is independent of R (we can choose U' so that $\mathrm{supp}(R) \subset U'$). This current is given by

$$\langle \tau, \Phi\rangle := c_\Phi.$$

This is the convergence in Theorem 1.1

Remark 4.2. If $(R_n) \subset \mathscr{C}(U)$ such that $R_n \leq cR$, $c \geq 0$, then $d^{-(k-p)n}(f^n)_* R_n$ converges to τ. Indeed, Proposition 4.1 implies that

$$\limsup \langle d^{-(k-p)n}(f^n)_* R_n, \Phi\rangle \leq c_\Phi$$

and

$$\limsup \langle d^{-(k-p)n}(f^n)_*(cR - R_n), \Phi\rangle \leq (c-1)c_\Phi.$$

Consider the sum of these inequalities. Since $\lim\langle d^{-(k-p)n}(f^n)_* R, \Phi\rangle = c_\Phi$, we deduce that $\lim\langle d^{-(k-p)n}(f^n)_* R_n, \Phi\rangle = c_\Phi$.

The following proposition shows that one needs to test only one form in order to check the convergence toward τ.

Proposition 4.3. *Let Φ be a real smooth $(k-p,k-p)$-form on U such that $dd^c\Phi$ is strictly positive. Let (R_{n_i}) be a sequence of currents in $\mathscr{C}(U)$. Then $d^{-n_i}(f^{n_i})_*R_{n_i}$ converges to τ if and only if $\lim\langle d^{-n_i}(f^{n_i})_*R_{n_i},\Phi\rangle=\langle\tau,\Phi\rangle$.*

Proof. Let Ψ be a real smooth $(k-p,k-p)$-form with compact support in U. Let $c>0$ such that $dd^c(c\Phi\pm\Psi)\geq 0$. Proposition 4.1 implies that

$$\limsup_{n\to\infty}\langle d^{-n_i}(f^{n_i})_*R_{n_i},c\Phi\pm\Psi\rangle\leq\langle\tau,c\Phi\pm\Psi\rangle.$$

If $\lim\langle d^{-n_i}(f^{n_i})_*R_{n_i},\Phi\rangle=\langle\tau,\Phi\rangle$ then $\lim\langle d^{-n_i}(f^{n_i})_*R_{n_i},\Psi\rangle=\langle\tau,\Psi\rangle$ for every Ψ. The proposition follows. $\square$

We have the following Corollary.

Corollary 4.4. *Let L' be a generic projective space of dimension $k-p$, close enough to L. Then there exists an increasing sequence (n_i) of integers such that $d^{-n_i}(f^{n_i})_*[L']\to\tau$. In particular, τ is weakly laminar.*

Proof. Let ρ be a smooth function as above such that $\rho>0$ on a neighbourhood of $\mathrm{id}\in\mathrm{Aut}(\mathbb{P}^k)$. We want to prove the convergence for $L':=A(L)$ and for ρ-almost every $A\in\mathrm{Aut}(\mathbb{P}^k)$. Let Φ be as in Proposition 4.3. We have seen that

$$\lim_{n\to\infty}\langle d^{-(k-p)n}(f^n)_*R_0,\Phi\rangle=c_\Phi.$$

Since $R_0=\int[A(L)]d\rho(A)$, we have

$$\lim_{n\to\infty}\int\langle d^{-(k-p)n}(f^n)_*[A(L)],\Phi\rangle d\rho(A)=c_\Phi.$$

On the other hand, the integrals $\langle d^{-(k-p)n}(f^n)_*[A(L)],\Phi\rangle$ are unifomly bounded with respect to n and A, and

$$\limsup_{n\to\infty}\langle d^{-(k-p)n}(f^n)_*[A(L)],\Phi\rangle\leq c_\Phi.$$

Therefore, for ρ-almost every A there exists (n_i) such that

$$\lim_{i\to\infty}\langle d^{-n_i}(f^{n_i})_*[L'],\Phi\rangle=c_\Phi,\quad\text{where } L':=A(L).$$

This, together with Proposition 4.3, implies that $d^{-n_i}(f^{n_i})_*[L']\to\tau$.

Let L_i be holomorphic images of $\mathbb{P}^{k-p}$ in $\mathbb{P}^k$ and let c_i be positive numbers. Assume that the sequence of currents $c_i[L_i]$ converges. Theorem 5.1 in [9] implies that the limit is a weakly laminar current. Then τ is weakly laminar. $\square$

The following proposition completes the proof of Theorem 1.1.

Proposition 4.5. *The current τ is extremal in $\mathscr{D}$. In particular, it is extremal in the cone of invariant positive closed (p,p)-currents supported on $\mathscr{A}$.*

Proof. Recall that $\mathscr{D}$ is a convex compact set. Assume that $\tau = \lambda S_1 + (1-\lambda)S_2$ with S_1 and S_2 in $\mathscr{D}$. Then Proposition 4.1 implies that $\langle S_1, \Phi\rangle \leq c_\Phi$ and $\langle S_2, \Phi\rangle \leq c_\Phi$. We also have

$$c_\Phi = \langle \tau, \Phi\rangle = \lambda\langle S_1, \Phi\rangle + (1-\lambda)\langle S_2, \Phi\rangle.$$

Hence $\langle S_1, \Phi\rangle = \langle S_2, \Phi\rangle = c_\Phi$. It follows that $S_1 = S_2 = \tau$. $\square$

We have the following version of Theorem 1.1 where the form R' is continuous but not necessarily positive and closed.

Theorem 4.6. *Let S be a positive closed (p,p)-current in $\mathscr{C}(U)$. Let (n_i) be an increasing sequence of integers such that $d^{-(k-p)n_i}(f^{n_i})_*S$ converges to τ. If φ is a continuous function on U, then $d^{-(k-p)n_i}(f^{n_i})_*(\varphi S)$ converges weakly to $c_{\varphi S}\tau$, where $c_{\varphi S} := \langle S \wedge T^{k-p}, \varphi\rangle$. In particular, $d^{-(k-p)n}(f^n)_*(\varphi\tau)$ converges weakly to $c_{\varphi\tau}\tau$. If R is as in Theorem 1.1 and R' is a continuous real (p,p)-form, not necessarily positive and closed, such that $-cR \leq R' \leq cR$, $c > 0$, then $d^{-(k-p)n}(f^n)_*R'$ converges weakly to $c_{R'}\tau$, where $c_{R'} := \langle T^{k-p}, R'\rangle$.*

We first prove the following proposition which can be easily extended to the case of meromorphic maps or correspondences, see also [2, 32] and [9, Remarque 5.9].

Proposition 4.7. *Let φ be a real-valued smooth function and let S be a positive closed (m,m)-current on $\mathbb{P}^k$, $0 \leq m \leq k-1$. Then*

$$\|\mathrm{dd^c}\big(d^{-(k-m)n}(f^n)_*(\varphi S)\big)\| = O(d^{-n}) \quad \textit{and} \quad \|\mathrm{d}\big(d^{-(k-m)n}(f^n)_*(\varphi S)\big)\| = O(d^{-n/2}).$$

In particular, if R' is a smooth (m,m)-form on $\mathbb{P}^k$ then

$$\|\mathrm{dd^c}\big(d^{-(k-m)n}(f^n)_*R'\big)\| = O(d^{-n}) \quad \textit{and} \quad \|\mathrm{d}\big(d^{-(k-m)n}(f^n)_*R'\big)\| = O(d^{-n/2}).$$

Proof. Observe that the first estimate is classical. We can write $\mathrm{dd^c}\varphi$ as a linear combination of positive closed $(1,1)$-forms. Therefore, $\mathrm{dd^c}\varphi \wedge S$ can be written as a linear combination of positive closed $(m+1,m+1)$-currents. On the other hand, we have seen in the Introduction that if S' is a positive closed $(m+1,m+1)$-current, then $\|(f^n)_*S'\| = d^{(k-m-1)n}\|S'\|$. Hence

$$\|\mathrm{dd^c}\big(d^{-(k-m)n}(f^n)_*(\varphi S)\big)\| = \|\big(d^{-(k-m)n}(f^n)_*(\mathrm{dd^c}\varphi \wedge S)\big)\| = O(d^{-n}).$$

For the second estimate in the proposition, we use, as in [9, 15], the product map $F : \mathbb{P}^k \times \mathbb{P}^k \to \mathbb{P}^k \times \mathbb{P}^k$ with $F(x,y) = (f(x), f(y))$. A simple computation on cohomology implies that $\|(F^n)_*\widetilde{S}\| \lesssim d^{(2k-s)n}$ for every positive closed (s,s)-current $\widetilde{S}$ in $\mathbb{P}^k \times \mathbb{P}^k$. We will show that

$$\|\partial\big(d^{-(k-m)n}(f^n)_*(\varphi S)\big)\| = O(d^{-n/2}).$$

We also need an analogous inequality with $\overline{\partial}$ instead of ∂, which is proved in the same way. Let Φ be a smooth $(k-m-1, k-m)$-form on $\mathbb{P}^k$ such that $\|\Phi\|_\infty = 1$. It is sufficient to check that

$$|\langle (f^n)_*(\partial\varphi \wedge S), \Phi\rangle| = |\langle \partial (f^n)_*(\varphi S), \Phi\rangle| \leq c d^{(k-m-\frac{1}{2})n},$$

where $c > 0$ is independent of Φ.

Consider the following forms and positive closed current in $\mathbb{P}^k \times \mathbb{P}^k$

$$\widehat{\phi} := i\partial\varphi \otimes \overline{\partial}\varphi, \quad \widehat{S} = S \otimes S \quad \text{and} \quad \widehat{\Phi} = \Phi \otimes \overline{\Phi}.$$

Then the desired inequality is equivalent to

$$|\langle (F^n)_*(\widehat{\phi} \wedge \widehat{S}), \widehat{\Phi}\rangle| \leq c^2 d^{(2k-2m-1)n}.$$

Since $\widehat{\phi}$ is smooth, it can be written as a linear combination of positive $(1,1)$-forms which are bounded by closed ones. Then $\widehat{\phi} \wedge \widehat{S}$ can be bounded by a positive closed $(2m+1, 2m+1)$-current. The last inequality follows from the previous observation on the mass of $(F^n)_*(\widetilde{S})$.

When R' is smooth, it can be written as a linear combination of forms of type φR, where φ is a smooth function and R is a positive closed (m,m)-form. The rest of the proposition follows.

Finally, note that in the proposition, one can replace $dd^c\phi$, $d\phi$, $dd^c R$, dR by continuous forms not necessarily closed. □

Proof of Theorem 4.6. Replacing φ by $\varphi + A$, $A > 0$, we can assume that φ is strictly positive. Since φ can be uniformly approximated by smooth functions, we can also assume that φ is smooth.

Let τ' be the limit of $d^{-(k-p)m_i}(f^{m_i})_*(\varphi S)$ for a subsequence (m_i) of (n_i). Observe that τ' is positive. By Proposition 4.7, τ' is closed. We also have

$$\begin{aligned} \|\tau'\| &= \int \tau' \wedge \omega_{\mathrm{FS}}^{k-p} = \lim_{i\to\infty} \int d^{-(k-p)m_i}(f^{m_i})_*(\varphi S) \wedge \omega_{\mathrm{FS}}^{k-p} \\ &= \lim_{i\to\infty} \int \varphi S \wedge d^{-(k-p)m_i}(f^{m_i})^*\omega_{\mathrm{FS}}^{k-p} = \int \varphi S \wedge T^{k-p} = c_{\varphi S}, \end{aligned}$$

see [32] for the last limit. Moreover, if τ'_n is a limit value of $d^{-(k-p)(m_i-n)}(f^{m_i-n})_*(\varphi S)$ then τ'_n is closed and $\tau' = d^{-(k-p)n}(f^n)_*\tau'_n$. It follows that $c_{\varphi S}^{-1}\tau'$ is an element of $\mathscr{D}$. On the other hand, we have $\tau' \leq \lim d^{-(k-p)n_i}(f^{n_i})_*(cS) = c\tau$ for $c > 0$ such that $\varphi \leq c$. Proposition 4.5 implies that $\tau' = c_{\varphi S}\tau$. Therefore, $d^{-(k-p)n_i}(f^{n_i})_*(\varphi S)$ converges to $c_{\varphi S}\tau$.

The assertion on $\varphi\tau$ is clear.

Arguing as in Lemma 3.1, we can regularize R and obtain a positive closed form $\widetilde{R}$ of mass 1 with compact support in U, strictly positive in a neighbourhood U'' of $\mathrm{supp}(R')$. Replace R by $\widetilde{R}$. Then, since R' can be uniformly approximated

by smooth forms in U'', we can assume that R' is smooth. We always have $-cR \leq R' \leq cR$ for some constant $c > 0$. Now, by Theorem 1.1, we can replace R' by $R' + cR$ and assume that R' is positive. Then the previous proof for φS is valid for R' in the present context. We use here Theorem 1.1 and the second part of Proposition 4.7. □

5 Equilibrium measure

In this section we prove Theorem 1.2. We have seen in Corollary 2.2 that the entropy of ν satisfies $h(\nu) \leq (k-p)\log d$. In order to prove the inequality $h(\nu) \geq (k-p)\log d$, we can follow the proof of Bedford-Smillie for Hénon maps [2], see also [34, 8]. For this purpose one only needs the following lemma.

Lemma 5.1. *Let L' be a generic projective space of dimension $k-p$, close enough to L. Then there exists an increasing sequence (n_i) of integers such that*

$$\frac{1}{n_i}\sum_{j=0}^{n_i-1} d^{-(k-p)n_i}(f^j)_*[L'] \wedge (f^{n_i-j})^*\omega_{\rm FS}^{k-p} \to \nu.$$

Proof. As in Corollary 4.4, we prove that if L' is generic,

$$\frac{1}{n_i}\sum_{j=0}^{n_i-1} d^{-(k-p)j}(f^j)_*[L'] \to \tau.$$

Then

$$\frac{1}{n_i}\sum_{j=0}^{n_i-1} d^{-(k-p)j}(f^j)_*[L'] \wedge T^{k-p} \to \nu,$$

see [6, 21, 32]. One only needs the following fact, applied to $S = d^{-(k-p)j}(f^j)_*[L']$ in order to complete the proof.

Write $d^{-n}(f^n)^*\omega_{\rm FS} = \omega_{\rm FS} + {\rm dd^c} g_n$, where g_n is a smooth function converging uniformly to the Green function g of f as $n \to \infty$, see e.g. [32]. We have $T = \omega_{\rm FS} + {\rm dd^c} g$. If S is a positive closed current of mass 1 and if Φ is a fixed smooth test form of right degree, it is easy to show that

$$\langle S \wedge d^{-(k-p)n}(f^n)^*\omega_{\rm FS}^{k-p}, \Phi\rangle - \langle S \wedge T^{k-p}, \Phi\rangle$$

converges to 0 uniformly on S, see [6, 21, 32]. □

Now, we have to prove the mixing. We follows the ideas of [2, 32], see [15] for another approach. Let φ and ψ be real-valued smooth functions. We want to check that

$$\langle \nu, \varphi(\psi \circ f^n)\rangle \to \langle \nu, \varphi\rangle\langle \nu, \psi\rangle.$$

We have

$$\langle \nu, \varphi(\psi \circ f^n) \rangle = \langle \varphi\tau \wedge T^{k-p}, \psi \circ f^n \rangle = \langle (f^n)_*(\varphi\tau \wedge T^{k-p}), \psi \rangle.$$

Define $c_\varphi := \langle \nu, \varphi \rangle$. Then the following proposition, applied to $m = k-p$, implies the result.

Proposition 5.2. *For all $0 \leq m \leq k-p$, the current $d^{-(k-p-m)n}(f^n)_*(\varphi\tau \wedge T^m)$ converges weakly to $c_\varphi \tau \wedge T^m$ as $n \to \infty$.*

Proof. The proof uses an induction on m. For $m = 0$ the proposition is deduced from Theorem 4.6. Assume the proposition for m, $0 \leq m \leq k-p-1$. Let Ψ be a smooth test form of right bidegree. Define $S' := \varphi\tau \wedge T^m$. We have

$$\begin{aligned}
&d^{-(k-p-m-1)n}\langle (f^n)_*(\varphi\tau \wedge T^{m+1}), \Psi \rangle \\
&\quad = d^{-(k-p-m-1)n}\langle S' \wedge T, (f^n)^*\Psi \rangle \\
&\quad = \lim_{s\to\infty} d^{-(k-p-m-1)n}\langle S' \wedge d^{-n-s}(f^{n+s})^*\omega_{\mathrm{FS}}, (f^n)^*\Psi \rangle \\
&\quad = \lim_{s\to\infty} d^{-(k-p-m-1)n}\langle S', d^{-n-s}(f^{n+s})^*\omega_{\mathrm{FS}} \wedge (f^n)^*\Psi \rangle \\
&\quad = \lim_{s\to\infty} d^{-(k-p-m)n}\langle (f^n)_*S', d^{-s}(f^s)^*\omega_{\mathrm{FS}} \wedge \Psi \rangle \\
&\quad = \lim_{s\to\infty} d^{-(k-p-m)n}\langle (f^n)_*S', (\omega_{\mathrm{FS}} + dd^c g_s) \wedge \Psi \rangle
\end{aligned}$$

The last integral is equal to

$$\begin{aligned}
&d^{-(k-p-m)n}\langle (f^n)_*S', \Psi \wedge \omega_{\mathrm{FS}} + g_s dd^c\Psi \rangle + d^{-(k-p-m)n}\langle dd^c[(f^n)_*S'], g_s\Psi \rangle \\
&\qquad + d^{-(k-p-m)n}\langle d[(f^n)_*S'], g_s d^c\Psi \rangle - d^{-(k-p-m)n}\langle d^c[(f^n)_*S'], g_s d\Psi \rangle.
\end{aligned}$$

Since the g_s are uniformly bounded, Proposition 4.7 implies that the last three terms tend to 0 when n and s tend to infinity. Since g_s converges uniformly to g, the induction hypothesis implies that the first term converges to

$$c_\varphi \langle \tau \wedge T^m, \Psi \wedge \omega_{\mathrm{FS}} + g dd^c\Psi \rangle = c_\varphi \langle \omega_{\mathrm{FS}} \wedge \tau \wedge T^m + dd^c(g\tau \wedge T^m), \Psi \rangle = c_\varphi \langle \tau \wedge T^{m+1}, \Phi \rangle$$

since $\omega_{\mathrm{FS}} + dd^c g = T$. This completes the proof. $\square$

6 Remarks and questions

We consider the case where $p = 1$ and $\mathbb{P}^k \setminus U$ is convex in $\mathbb{P}^k \setminus L \simeq \mathbb{C}^k$. The fact that τ is a current of bidegree $(1, 1)$ allows us to prove some further properties of the attracting set $\mathscr{A}$.

The following proposition shows that for a generic point in $\mathbb{P}^k$ most of its preimages are out of U.

Proposition 6.1. *Let* Leb *denote the Lebesgue measure in* $\mathbb{P}^k$. *Then* $\|(f^n)^*\mathrm{Leb}\|_U = o(d^n)$. *In particular, for every* $\epsilon > 0$

$$\lim_{n\to\infty} \mathrm{Leb}\{a \in \mathbb{P}^k,\ \#f^{-n}(a) \cap U \geq \epsilon d^n\} = 0.$$

Proof. Observe that U admits a neighbourhood U'' satisfying the same properties, i.e. $f(U'') \Subset U''$ and $\mathbb{P}^k \setminus U''$ is convex. We replace U by U''. Then we can replace U in the proposition by a compact set $K \subset U$.

Let S be a smooth $(k-1, k-1)$-form on $\mathbb{P}^k \setminus I$ such that $\mathrm{dd^c} S = \mathrm{Leb}$. Here, we identify Leb with a volume form. Let φ be a positive smooth function with support in U such that $\varphi = 1$ on K.

Observe that since $\mathbb{P}^k \setminus U$ is convex, we can find a smooth form $R \in \mathscr{C}(U)$ strictly positive on the support of $\mathrm{dd^c}\varphi$. Theorem 4.6 applied to $R' := \mathrm{dd^c}\varphi$ and to R yields $d^{-n}(f^n)_* R' \to 0$. Hence

$$\langle d^{-n}(f^n)^*\mathrm{Leb}, \varphi\rangle = \langle d^{-n}\mathrm{dd^c}(f^n)^* S, \varphi\rangle = \langle S, d^{-n}(f^n)_* R'\rangle \to 0.$$

The first assertion follows.

Now observe that $\#f^{-n}(a) \cap K \geq \epsilon d^n$ if and only if $(f^n)_* \mathbf{1}_K(a) \geq \epsilon d^n$. Hence

$$\begin{aligned} &\epsilon d^n \mathrm{Leb}\{a \in \mathbb{P}^k,\ \#f^{-n}(a) \cap K \geq \epsilon d^n\} \\ &\leq\ \langle \mathrm{Leb}, (f^n)_* \mathbf{1}_K\rangle = \langle (f^n)^*\mathrm{Leb}, \mathbf{1}_K\rangle = \|(f^n)^*\mathrm{Leb}\|_K. \end{aligned}$$

We conclude using the first assertion in the proposition. □

Let R be a positive closed $(1,1)$-current of mass 1 in U. We call *canonical quasi-potential* of R the unique quasi-psh function g_R on $\mathbb{P}^k$ such that $g_R(I) = 0$ and $\mathrm{dd^c} g_R = R - \omega_{\mathrm{FS}}$. Denote by g_τ the canonical quasi-potential of τ. The following result gives a refined version of Theorem 1.1 for the bidegree $(1,1)$ case.

Theorem 6.2. *Let* S *be an element of* $\mathscr{D}$ *and* g_S *its canonical quasi-potential. Then* $g_S \leq g_\tau$ *on* $\mathbb{P}^k$ *and* $g_S = g_\tau$ *on* U. *If the jacobian of* f *satisfies* $|\mathrm{Jac}(f)| < 1$ *on* U *then* τ *is the unique invariant positive closed* $(1,1)$*-current with mass* 1 *supported in* U.

Proof. We prove that $g_S = g_\tau$ in $\mathbb{P}^k \setminus \overline{U}$. Consider a $(k-1,k)$-form Ψ with support in $\mathbb{P}^k \setminus \overline{U}$. We first show that $\langle g_S - g_\tau, \partial\Psi\rangle = 0$. By Hodge theory, $\partial\Psi$ is $\mathrm{dd^c}$-exact in $\mathbb{P}^k$. Write $\Psi = \mathrm{dd^c}\Phi$. We have $\langle g_S - g_\tau, \partial\Psi\rangle = \langle S - \tau, \Phi\rangle$. Since $\mathrm{dd^c}\Phi = 0$ on U, Proposition 4.1 implies that the last integral vanishes. Hence, $\partial(g_S - g_\tau) = 0$ on $\mathbb{P}^k \setminus \overline{U}$. Since $g_S - g_\tau$ is real-valued, $g_S - g_\tau$ is constant in $\mathbb{P}^k \setminus \overline{U}$. This constant is zero since $g_S(I) = g_\tau(I) = 0$. Since S and τ have supports in $\mathscr{A}$, g_S and g_τ are pluriharmonic on $\mathbb{P}^k \setminus \mathscr{A}$. We deduce that $g_S = g_\tau$ on the connected component W of $\mathbb{P}^k \setminus \mathscr{A}$ which contains $\mathbb{P}^k \setminus U$.

Let Ω be a smooth positive (k,k)-form with support in a small neighbourhood of $\mathscr{A}$. We show that $\langle g_S - g_\tau, \Omega\rangle$ is negative. There exists a smooth real $(k-$

$1,k-1)$-form Φ on U such that $dd^c\Phi = \Omega$. Since $g_S - g_\tau = 0$ on W, we have $\langle g_S - g_\tau, \Omega\rangle = \langle g_S - g_\tau, dd^c\Phi\rangle = \langle S - \tau, \Phi\rangle$. Since $dd^c\Phi$ is positive, Proposition 4.1 implies that the last integral is negative. Hence, $g_S \leq g_\tau$.

Define $\mathscr{A}' := \mathbb{P}^k \setminus W$. We show that $f(\mathscr{A}') \subset \mathscr{A}'$. We have $f(\mathscr{A}') \Subset U$ and $\partial\mathscr{A}' \subset \mathscr{A} \subset \mathscr{A}'$. Since f is open, $\partial f(\mathscr{A}') \subset f(\partial\mathscr{A}') \subset f(\mathscr{A}) \subset \mathscr{A}$ and then $f(\mathscr{A}') \cap W = \varnothing$. We deduce from the definition of W that $f(\mathscr{A}') \subset \mathscr{A}'$. If the jacobian of f satisfies $|\mathrm{Jac}(f)| < 1$ on U then $\mathscr{A}'$ has Lebesgue measure zero and $g_S = g_\tau$ almost everywhere. Therefore $g_S = g_\tau$ everywhere since they are quasi-psh, and $\mathscr{D}$ is reduced to $\{\tau\}$. □

Remark 6.3. The hypothesis in Theorem 6.2 is stable under small pertubations on f. In particular, small pertubations of polynomial maps give us examples satisfying this condition. For such a map, since the sum of Lyapounov exponents associated to ν is equal to $\langle \nu, \log|\mathrm{Jac}(f)|\rangle$ which is negative, ν admits a strictly negative Lyapounov exponent. In the dimension 2 case, de Thélin proved that ν has a strictly positive exponent [8]; then ν is hyperbolic.

The following questions are still open for the general case.

Questions (with de Thélin). 1. Is τ the unique (invariant) positive closed (p,p)-current of mass 1 with support in $\mathscr{A}$? Is $\mathscr{A}$ always of Lebesgue measure zero ? Is the quasi-potential of τ unbounded in the bidegree $(1,1)$ case ?

2. Is ν the unique measure of maximal entropy on $\mathscr{A}$? Is it exponentially mixing and singular with respect to the Lebesgue measure ?

3. Is T laminar in U ? Does ν have a product structure ? Are periodic saddle points equidistributed on the support of ν ?

4. Does ν have strictly negative Lyapounov exponents ? Even when we assume that, the previous questions are open.

A Slicing of horizontal currents

In this appendix we give some results on the slicing of currents used in this paper.

Consider the open set $U \subset \mathbb{P}^k$ as in the Introduction, the manifold V of dimension m and the canonical projections $\pi_V : V \times U \to V$, $\pi_U : V \times U \to U$ as in Section 3. Let $\mathscr{R}$ be a positive closed (p,p)-current on $V \times U$. The slicing theory [17, 26] allows us to define slices $\langle \mathscr{R}, \pi_V, \theta\rangle$ for almost every $\theta \in V$, see also [15]. We can consider $\langle \mathscr{R}, \pi_V, \theta\rangle$ as the intersections of $\mathscr{R}$ with the currents of integration on $\pi_V^{-1}(\theta)$. They are positive closed (p,p)-currents on $\{\theta\} \times U$. We often identify them to (p,p)-currents in U or consider them as currents on $V \times U$, of bidegree $(p+m, p+m)$.

Slicing is the generalization of restriction of forms to level sets of holomorphic maps. When $\mathscr{R}$ is a continuous form, $\langle \mathscr{R}, \pi_V, \theta\rangle$ is simply the restriction of $\mathscr{R}$ to $\pi_V^{-1}(\theta)$. When $\mathscr{R}$ is the current of integration on an analytic subset X of $V \times U$, $\langle \mathscr{R}, \pi_V, \theta\rangle$ is the current of integration on the analytic set $X \cap \pi_V^{-1}(\theta)$ for θ generic.

If Ω is a smooth form of maximal degree with compact support in V and if Ψ is a smooth $(k-p, k-p)$-form in $V \times U$, then

$$\int_V \langle \mathscr{R}, \pi_V, \theta \rangle(\Psi)\Omega(\theta) = \langle \mathscr{R} \wedge \pi_V^*(\Omega), \Psi \rangle.$$

Here, we need that π_V is proper on $\mathrm{supp}(\Psi) \cap \mathrm{supp}(\mathscr{R})$. In particular, this holds when Ψ has compact support.

Let y denote the coordinates in a chart of V and λ_V the associated standard volume form. Let $\psi(y)$ be a positive smooth function with compact support such that $\int \psi \lambda_V = 1$. Define $\psi_\epsilon(y) := \epsilon^{-2m}\psi(\epsilon^{-1}y)$ and $\psi_{\theta,\epsilon}(y) := \psi_\epsilon(y-\theta)$ (the measures $\psi_{\theta,\epsilon}\lambda_V$ approximate the Dirac mass at θ). Then, for every smooth test $(k-p, k-p)$-form Ψ with compact support in $V \times U$ one has

$$\langle \mathscr{R}, \pi_V, \theta \rangle(\Psi) = \lim_{\epsilon \to 0} \langle \mathscr{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}\lambda_V), \Psi \rangle$$

when $\langle \mathscr{R}, \pi_V, \theta \rangle$ exists. This property holds for all choice of ψ and also for Ψ such that π_V is proper on $\mathrm{supp}(\Psi) \cap \mathrm{supp}(\mathscr{R})$. Conversely, when the previous limit exists and is independent of ψ, it defines $\langle \mathscr{R}, \pi_V, \theta \rangle$ and one says that $\langle \mathscr{R}, \pi_V, \theta \rangle$ *is well defined*.

Our main result of this section is the following proposition.

Proposition A.1. *Assume that $\mathscr{R}$ is horizontal, i.e. $\pi_U(\mathrm{supp}(\mathscr{R})) \Subset U$. Then the slice $\langle \mathscr{R}, \pi_V, \theta \rangle$ exists for every $\theta \in V$. Moreover, if Ψ is a real smooth $(k-p, k-p)$-form on $V \times U$ such that $\mathrm{dd^c}\Psi \geq 0$ then $\langle \mathscr{R}, \pi_V, \theta \rangle(\Psi)$ defines a psh function on $\theta \in V$. If $\mathrm{dd^c}\Psi = 0$ then $\langle \mathscr{R}, \pi_V, \theta \rangle(\Psi)$ is pluriharmonic.*

Proof. Since the problem is local, we can assume that V is a ball in $\mathbb{C}^m$ and use the above notation. Let Ψ be a real smooth $(k-p, k-p)$-form on $V \times U$. We want to prove that

$$\lim_{\epsilon \to 0} \langle \mathscr{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}\lambda_V), \Psi \rangle$$

exists and does not depend on the choice of ψ. An analogous result as Lemma 3.1 allows us to assume that $\mathrm{dd^c}\Psi \geq 0$. Since π_V is proper on the support of $\mathscr{R}$, the current $\varphi := (\pi_V)_*(\mathscr{R} \wedge \Psi)$ is well defined and has bidegree $(0,0)$. We also have $\mathrm{dd^c}\varphi = (\pi_V)_*(\mathscr{R} \wedge \mathrm{dd^c}\Psi) \geq 0$. Then φ is (equal to) a psh function on V. We have

$$\lim_{\epsilon \to 0} \langle \mathscr{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}\lambda_V), \Psi \rangle = \lim_{\epsilon \to 0} \int_V \varphi \psi_{\theta,\epsilon}\lambda_V.$$

A classical property of psh functions implies that the last limit is equal to $\varphi(\theta)$. The proposition follows. $\square$

Remark A.2. If Φ is a real smooth $(k-p, k-p)$-form on U such that $\mathrm{dd^c}\Phi \geq 0$, then $\langle \mathscr{R}, \pi_V, \theta \rangle(\Phi)$ is also psh on V. Indeed, we can apply Proposition A.1 to $\Psi := \pi_U^*(\Phi)$.

Proposition A.3. *Under the hypothesis of Proposition A.1, the mass of $\langle \mathscr{R}, \pi_V, \theta\rangle$ is independent of θ.*

Proof. Consider the positive closed form $\Phi := \omega_{\rm FS}^{k-p}$. Define as above $\Psi := \pi_U^*(\Phi)$ and $\varphi := (\pi_V)_*(\mathscr{R}\wedge\Psi)$. Then φ is closed. It follows that φ is a constant function. Hence the mass of $\langle \mathscr{R}, \pi_V, \theta\rangle$, which is equal to $\langle \mathscr{R}, \pi_V, \theta\rangle(\Phi) = \varphi(\theta)$, does not depend on θ. $\square$

Acknowledgments. I would like to thank Henry de Thélin, Viet-Anh Nguyen and Nessim Sibony for their remarks.

References

[1] Bedford E., Lyubich M., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$. IV. The measure of maximal entropy and laminar current, *Invent. Math.*, **112** (1993), 77-125.

[2] Bedford E., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$ III: Ergodicity, exponents and entropy of the equilibrium measure, *Math. Ann.*, **294** (1992), 395-420.

[3] Bonifant A., Dabija M., Milnor J., Elliptic curves as attractors in $\mathbb{P}^2$ Part 1 : Dynamics, *preprint*, arXiv:math.DS/0601015

[4] Bowen R., Topological entropy for non compact sets, *Trans. A.M.S.*, **184** (1973), 125-136.

[5] Briend J.-Y., Duval J., Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de $\mathbb{CP}^k$, *Acta Math.*, **182** (1999), 143-157.

[6] Demailly J.-P., Monge-Ampère Operators, Lelong numbers and Intersection theory in Complex Analysis and Geometry, *Plemum Press* (1993), 115-193, *(V. Ancona and A. Silva editors)*.

[7] de Thélin H., Un phénomène de concentration de genre, *Math. Ann.*, **332** (2005), no. 3, 483-498.

[8] —–, Sur la construction de mesures selles, *Ann. Inst. Fourier*, to appear.

[9] Dinh T.-C., Suites d'applications méromorphes multivaluées et courants laminaires, *J. Geom. Analysis*, **15** (2005), 207-227.

[10] Dinh T.-C., Sibony N., Dynamique des applications d'allure polynomiale, *J. Math. Pures Appl.*, **82** (2003), 367-423.

[11] —–, Une borne supérieure pour l'entropie topologique d'une application rationnelle, *Ann. of Math.*, **161** (2005), 1637-1644.

[12] —–, Regularization of currents and entropy, *Ann. Sci. Ecole Norm. Sup.*, **37** (2004), 959-971.

[13] —–, Green currents for holomorphic automorphisms of compact Kähler manifolds, *J. Amer. Math. Soc.*, **18** (2004), 291-312.

[14] —–, Distribution des valeurs d'une suite de transformations méromorphes et applications, *Comment. Math. Helv.*, **81** (2006), 221-258.

[15] —–, Geometry of currents, intersection theory and dynamics of horizontal-like maps, *Ann. Inst. Fourier*, to appear.

[16] Dujardin R., Laminar currents and birational dynamics, *Duke Math. J.*, **131** (2006), no. 2, 219-247.

[17] Federer H., *Geometric Measures Theory*, New York, Springer Verlag, 1969.

[18] Fornæss J.-E., Sibony N., Complex dynamics in higher dimensions, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, **439**, *Complex potential theory (Montreal, PQ, 1993)*, 131-186, Kluwer Acad. Publ., Dordrecht, 1994.

[19] —–, Complex dynamics in higher dimension. II. Modern methods in complex analysis (Princeton, NJ, 1992), 135-182, *Ann. of Math. Stud.*, **137**, Princeton Univ. Press, Princeton, NJ, 1995.

[20] —–, Dynamics of $\mathbb{P}^2$ (examples). Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), 47-85, *Contemp. Math.*, **269**, Amer. Math. Soc., Providence, RI, 2001.

[21] —–, Oka's inequality for currents and applications, *Math. Ann.*, **301** (1995), 399-419.

[22] Fornæss J.-E., Weickert B., Attractors in $\mathbb{P}^2$, *Several complex variables (Berkeley, CA, 1995–1996)*, 297-307, *Math. Sci. Res. Inst. Publ.*, **37**, Cambridge Univ. Press, Cambridge, 1999.

[23] Gromov M., On the entropy of holomorphic maps, *Enseignement Math.*, **49** (2003), 217-235. *Manuscript* (1977).

[24] Jonsson M., Weickert B., A nonalgebraic attractor in $\mathbf{P}^2$, *Proc. Amer. Math. Soc.*, **128** (2000), no. 10, 2999-3002.

[25] Harvey R., Polking J., Extending analytic objects, *Comm. Pure Appl. Math.*, **28** (1975), 701-727.

[26] Harvey R., Shiffman B., A characterization of holomorphic chains, *Ann. of Math. (2)*, **99** (1974), 553-587.

[27] Hörmander L., *The analysis of Linear partial differential operators I*, Springer-Verlag, 1983.

[28] Katok A., Hasselblatt B., *Introduction to the modern theory of dynamical systems*, Cambridge Univ., Press. Encycl. of Math. and its Appl. **54**, 1995.

[29] Lelong P., *Fonctions plurisousharmoniques et formes différentielles positives*, Dunod Paris, 1968.

[30] Rong F., Critically finite maps and attractors on $\mathbb{P}^k$, *preprint*, 2005. arXiv:math.DS/0510650

[31] Russakovskii A., Shiffman B., Value distribution for sequences of rational mappings and complex dynamics, *Indiana Univ. Math. J.*, **46** (1997), 897–932.

[32] Sibony N., Dynamique des applications rationnelles de $\mathbb{P}^k$, *Panoramas et Synthèses*, **8** (1999), 97-185.

[33] Walters P., *An introduction to ergodic theory*, Berlin-Heidelberg-NewYork, Springer 1982.

[34] Yomdin Y., Volume growth and entropy, *Israel J. Math.*, **57** (1987), 285-300.

Tien-Cuong Dinh
Institut de Mathématique de Jussieu
Plateau 7D, Analyse Complexe
175 rue du Chevaleret
75013 Paris, France
dinh@math.jussieu.fr